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OPTIMAL COMPUTING FORMS FOR THE TWO-BODY C AND S SERIES

by C. R. Herron, E. R. Lancaster, and W. R. Trebilcock

Goddard Space Flight Center
Greenbelt, Md.

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ABSTRACT

The recently developed universal form of the two-body problem involves several transcendental functions. Since these functions are evaluated so frequently, it is worthwhile to develop approximations that minimize the number of arithmetical operations required. This paper presents several such approximations, based on theories of Chebyshev and Knuth, with bounds for the errors incurred when using them.

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INTRODUCTION

The classical solutions of the two-body problem separate naturally into the three cases of elliptic, parabolic, and hyperbolic motion, the mathematics being considerably different for each case. A unified formulation is possible, valid for all three cases, if certain transcendental functions, which we call the C and S functions, are introduced.

The unified formulation is fully developed by Battin (Reference 1) and will not concern us here. The purpose of this paper is to present approximations for the C and S functions and their derivative functions, which reduce significantly the computation times required for their evaluation when compared to those required by Taylor series expansions.

THE C AND S FUNCTIONS

The C and S functions are defined by

$$S(x) = \left(x^{1/2} - \sin x^{1/2}\right)\Big/x^{3/2}, \qquad x > 0 \tag{1}$$

$$= \left[\sinh(-x)^{1/2} - (-x)^{1/2}\right]\Big/(-x)^{3/2}, \qquad x < 0\ ; \tag{2}$$

$$C(x) = \left(1 - \cos x^{1/2}\right)\Big/x, \qquad x > 0 \tag{3}$$

$$= \left[1 - \cosh(-x)^{1/2}\right]\Big/x, \qquad x < 0\ . \tag{4}$$

Since these functions are indeterminate for $x = 0$ and present accuracy problems when evaluated in the neighborhood of $x = 0$, it is natural to replace the above forms by the following series, convergent for all values of x:

$$S(x) = \sum_{i=0}^{\infty} \frac{(-x)^i}{(2i+3)!}, \tag{5}$$

For large values of x, the convergence of these series will be slow. It is then convenient to use the following reduction formulas, easily derived from Equations 1 through 4:

$$A(x) = 1 - xS(x) , \tag{7}$$

$$2C(4x) = [A(x)]^2 , \tag{8}$$

$$4S(4x) = S(x) + A(x)C(x) . \tag{9}$$

THE C' AND S' FUNCTIONS

The derivatives $S'(x)$ and $C'(x)$ are needed for certain problems of orbit determination, guidance, and optimization. From Equations 1 through 4 we obtain

$$S'(x) = [C(x) - 3S(x)]/2x ,$$

$$C'(x) = [A(x) - 2C(x)]/2x .$$

These forms suffer accuracy problems in the neighborhood of $x = 0$, again forcing us to series representations. Differentiating Equations 5 and 6, we have

$$S'(x) = \sum_{i=1}^{\infty} \frac{i(-x)^i}{(2i+3)!} , \tag{10}$$

$$C'(x) = \sum_{i=1}^{\infty} \frac{i(-x)^i}{(2i+2)!} , \tag{11}$$

convergent for all values of x.

For large values of x, the following reduction formulas (obtained by differentiating Equations 7, 8, and 9) are useful.

$$B(x) = S(x) + xS'(x) , \tag{12}$$

$$C'(4x) = -A(x)B(x) , \tag{13}$$

$$4S'(4x) = S'(x) + A(x)C'(x) - B(x)C(x) . \tag{14}$$

THE FIKE-KNUTH ALGORITHM

Our first step in obtaining economical computing forms for Equations 5, 6, 10, and 11 was the construction of sixth degree polynomial approximations on various intervals in the sense of Chebyshev. In other words, these polynomials minimize the magnitude of the maximum error on the interval. The program to accomplish this was written by the third author, based on ideas of Stoer (Reference 2). The coefficients of these polynomials are given in the section entitled Numerical Results.

Assume that the approximating polynomial has the form

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 . \tag{15}$$

The evaluation of Equation 15 by the usual method of nested multiplication requires six multiplications and six additions. However, by using recently developed polynomial evaluation methods (References 3 and 4), Equation 15 can be evaluated with four multiplications and seven additions. The form and parameters for the algorithm, as it applies to our functions, are given in the Numerical Results section.

In the following description of the algorithm, a_6 is assumed to be positive. If a_6 is negative, a minor change is necessary.

Fike's modification of Knuth's method begins with a conversion: let $\mu = \sqrt[6]{a_6}$, and let $c_k = a_k/\mu^k$ for $k = 0, 1, \cdots, 5$. Then compute

$$\begin{aligned} p &= \tfrac{1}{2}(c_5 - 1) , & D'' &= c_2 - pC' , \\ B' &= c_4 - p(p+1) , & E' &= 2D' - B' + 1 , \\ C' &= c_3 - pB' , & E'' &= 2D'' - B'D' - C' , \\ D' &= p - B' , & E''' &= c_1 - B'D'' . \end{aligned}$$

Find a real root q of the cubic equation*

$$2q^3 + E'q^2 + E''q + E''' = 0 \tag{16}$$

*See Appendix A.

and compute

$$A = \frac{1}{2}B' - q ,$$

$$C = p - 2A ,$$

$$B = q - 2AC - A^2 ,$$

$$D = C' - q(1 + D') - q^2 - D'' - A^2(1 + C) - BC ,$$

$$E = q^2 + qD' + D'' - (A^2 + B)C ,$$

$$F = c_0 - (q^2 + qD' + D'')\left[C' - q(1 + D') - q^2 - D''\right] .$$

Then our polynomial can be evaluated as follows:

$$q_1 = \mu x ,$$

$$q_2 = (q_1 + A)^2 ,$$

$$q_3 = (q_2 + B)(q_1 + C) ,$$

$$P(x) = (q_2 + q_3 + D)(q_3 + E) + F .$$

In case $a_6 < 0$, let $T(x) = -P(x)$ and perform all the steps above, except the last, for $T(x)$. The last step should be

$$P(x) = -\left[T(x)\right] = (q_2 + q_3 + D)(-q_3 - E) - F .$$

If the machine being used has a "load negative" feature that is equivalent in execution time to "load positive," and if subtraction is likewise equivalent to addition, then this modification is equivalent to the original.

As Fike points out, his method is a slight variation of that of Knuth (Reference 4), and since Knuth's method was inspired by Motzkin (Reference 5), the three types bear a strong family resemblance. Each begins with a polynomial in the form of Equation 15 with $a_6 = 1$, and solves for the parameters in the final evaluation scheme by expanding the scheme into a sixth degree polynomial and equating its coefficients with those of Equation 15. To admit treatment of the general polynomial of degree six, however, some transformation must be made so that $a_6 = 1$. The most

straightforward way is making

$$Q(x) = P(x)/a_6$$

and then applying any of the three methods to $Q(x)$, adding an extra step at the last in multiplying the result by a_6. Fike specifies a different sort of transformation; his may be thought of as converting Equation 15 into

$$R(x) = x + \frac{a_5}{\mu^5} x^5 + \frac{a_4}{\mu^4} x^4 + \frac{a_3}{\mu^3} x^3 + \frac{a_2}{\mu^2} x^2 + \frac{a_1}{\mu} x + a_0 .$$

Again, any of the three methods apply to $R(x)$, and values of $P(x)$ are obtained by using μx in the scheme for $R(x)$, since $R(\mu x) = P(x)$.

This transformation, though a bit more complicated, is admirably suited to our particular problem. The type of polynomial with which we are dealing has the not uncommon characteristic that

$$|a_6| < |a_5| < \cdots < |a_0|$$

and, in addition, $|a_6|$ is very small. For example, suppose the coefficients of form (15) are

$$a_6 = + .11 \times 10^{-10} ,$$

$$a_5 = - .21 \times 10^{-8} ,$$

$$a_4 = + .28 \times 10^{-6} ,$$

$$a_3 = - .25 \times 10^{-4} ,$$

$$a_2 = + .14 \times 10^{-2} ,$$

$$a_1 = - .42 \times 10^{-1} ,$$

$$a_0 = + .50 .$$

If we use the division transformation, the coefficients b_i of $Q(x)$ are

$$b_6 = + 1.0 ,$$

$$b_5 = - .18 \times 10^3 ,$$

$$b_4 = +.24 \times 10^5 ,$$

$$b_3 = -.22 \times 10^7 ,$$

$$b_2 = +.12 \times 10^9 ,$$

$$b_1 = -.36 \times 10^{10} ,$$

$$b_0 = +.44 \times 10^{11} .$$

Here the errors in the numbers a_i have become greatly magnified; worse yet, the arithmetic of parameter production using the large numbers b_i is likely to suffer the effects of large error propagation. In contrast, Fike's transformation gives us

$$c_6 = 1 ,$$

$$c_5 = -.27 \times 10^1 ,$$

$$c_4 = +.54 \times 10^1 ,$$

$$c_3 = -.73 \times 10^1 ,$$

$$c_2 = +.62 \times 10^1 ,$$

$$c_1 = -.28 \times 10^1 ,$$

$$c_0 = .50 .$$

These numbers of manageable size lend themselves very well to whichever scheme we choose. For comparison, the two transformations above were evaluated by the Knuth algorithm for 40 points over the interval $[-1, +1]$, and the differences between these values and the true values of the polynomial were obtained. For the division transformation, the absolute value of the maximum error was $.92 \times 10^{-12}$; for the Fike transformation, this was $.16 \times 10^{-14}$, a reduction by a factor of more than 500. Several other test cases were run, with results that apparently verify the conclusion that the Fike transformation used on this type of polynomial has a very definite advantage. There are, of course, other transformations that produce polynomials in which $a_6 = 1$. In general, one should use the transformation that keeps the coefficients of the transformed polynomial as small as possible.

NUMERICAL RESULTS

The four approximation polynomials were generated for each of the intervals [- 1, + 1], [- 2, + 2], [- 4, + 4], [- 16, + 16], converted to Equation 15 and parameters for the Fike evaluation scheme were obtained. In each case, the values given by the final scheme were tested against "true" values of the original function for all multiples of .002 in the interval concerned. The true values came from expanding the power series of the function (1) for enough terms to guarantee that the relative error from truncation would be less than 10^{-15}. The following tables exhibit, for each of the sixteen functions considered, the coefficients a_i for Equation 15, the parameters A, B, C, D, E, and F for the Fike scheme, and the maximum absolute errors for both methods. For comparison, a degree 4 approximation polynomial was evaluated by both methods for the functions C(x) and S(x) on the interval [- 1, + 1], and the results are presented here also.

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$$C(X) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \quad \text{on } [-h, h]$$

$h = 1$

$a_0 = +0.4999999999999998 \times 10^{0}$
$a_1 = -0.4166666666667176 \times 10^{-1}$
$a_2 = +0.1388888888888999 \times 10^{-2}$
$a_3 = -0.2480158725995993 \times 10^{-4}$
$a_4 = +0.2755731917059028 \times 10^{-6}$
$a_5 = -0.2087759200397967 \times 10^{-8}$
$a_6 = +0.1147134108311665 \times 10^{-10}$

$M_a = 0.763 \times 10^{-15}$

$h = 2$

$a_0 = +0.4999999999999993 \times 10^{0}$
$a_1 = -0.4166666666700118 \times 10^{-1}$
$a_2 = +0.1388888888892785 \times 10^{-2}$
$a_3 = -0.2480158663241807 \times 10^{-4}$
$a_4 = +0.2755731881710992 \times 10^{-6}$
$a_5 = -0.2088010277268315 \times 10^{-8}$
$a_6 = +0.1147215380312168 \times 10^{-10}$

$M_a = 0.956 \times 10^{-13}$

$h = 4$

$a_0 = +0.4999999999998401 \times 10^{0}$
$a_1 = -0.4166666668808485 \times 10^{-1}$
$a_2 = +0.1388888889138842 \times 10^{-2}$
$a_3 = -0.2480157659289839 \times 10^{-4}$
$a_4 = +0.2755731272513377 \times 10^{-6}$
$a_5 = -0.2089014196095935 \times 10^{-8}$
$a_6 = +0.1147636934013430 \times 10^{-10}$

$M_a = 0.122 \times 10^{-10}$

$h = 16$

$a_0 = +0.4999999894793170 \times 10^{0}$
$a_1 = -0.4166675473500692 \times 10^{-1}$
$a_2 = +0.1388889916034137 \times 10^{-2}$
$a_3 = -0.2479883633119184 \times 10^{-4}$
$a_4 = +0.2755565077419917 \times 10^{-6}$
$a_5 = -0.2109148028487573 \times 10^{-8}$
$a_6 = +0.1156091702389399 \times 10^{-10}$

$M_a = 0.202 \times 10^{-6}$

$$q_1 = \sqrt[6]{a_6}\ x$$
$$q_2 = (q_1 + A)^2$$
$$q_3 = (q_2 + B)(q_1 + C)$$
$$C(X) = (q_2 + q_3 + D)(+q_3 + E) + F \quad \text{on } [-h, h]$$

$h = 1$

$A = +0.4513408582627891 \times 10^0$
$B = +0.3744865190483202 \times 10^1$
$C = -0.2769272800754423 \times 10^1$
$D = +0.9433565393074166 \times 10^1$
$E = +0.1055413968372178 \times 10^2$
$F = +0.6288190624578802 \times 10^{-2}$

$M = 0.583 \times 10^{-14}$

$h = 2$

$A = +0.4512008438957284 \times 10^0$
$B = +0.3743936586512442 \times 10^1$
$C = -0.2769076432349482 \times 10^1$
$D = +0.9429681327692907 \times 10^1$
$E = +0.1055053194443676 \times 10^2$
$F = +0.6284207351324604 \times 10^{-2}$

$M = 0.994 \times 10^{-13}$

$h = 4$

$A = +0.4507019590625572 \times 10^0$
$B = +0.3740448131455307 \times 10^1$
$C = -0.2768317205414987 \times 10^1$
$D = +0.9414899439055362 \times 10^1$
$E = +0.1053701311149719 \times 10^2$
$F = +0.6272339359526764 \times 10^{-2}$

$M = 0.122 \times 10^{-10}$

$h = 16$

$A = +0.4405766736959988 \times 10^0$
$B = +0.3669989101432331 \times 10^1$
$C = -0.2752824996097087 \times 10^1$
$D = +0.9117484138879304 \times 10^1$
$E = +0.1026464040151929 \times 10^2$
$F = +0.6161613003116790 \times 10^{-2}$

$M = 0.202 \times 10^{-6}$

$$S(X) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \qquad \text{on } [-h, h]$$

$h = 1$

$a_0 = +0.1666666666666665 \times 10^{0}$
$a_1 = -0.8333333333333568 \times 10^{-2}$
$a_2 = +0.1984126984129264 \times 10^{-3}$
$a_3 = -0.2755731919939401 \times 10^{-5}$
$a_4 = +0.2505210785999854 \times 10^{-7}$
$a_5 = -0.1605953765319026 \times 10^{-9}$
$a_6 = +0.7650283228592385 \times 10^{-12}$

$M_a = 0.555 \times 10^{-16}$

$h = 2$

$a_0 = +0.1666666666666663 \times 10^{0}$
$a_1 = -0.8333333333352921 \times 10^{-2}$
$a_2 = +0.1984126984129057 \times 10^{-3}$
$a_3 = -0.2755731883077317 \times 10^{-5}$
$a_4 = +0.2505210816170333 \times 10^{-7}$
$a_5 = -0.1606101133600677 \times 10^{-9}$
$a_6 = +0.7647926042737674 \times 10^{-12}$

$M_a = 0.566 \times 10^{-14}$

$h = 4$

$a_0 = +0.1666666666666581 \times 10^{0}$
$a_1 = -0.8333333334593103 \times 10^{-2}$
$a_2 = +0.1984126984258407 \times 10^{-3}$
$a_3 = -0.2755731292513204 \times 10^{-5}$
$a_4 = +0.2505210496761327 \times 10^{-7}$
$a_5 = -0.1606691704048320 \times 10^{-9}$
$a_6 = +0.7650122280184766 \times 10^{-12}$

$M_a = 0.720 \times 10^{-12}$

$h = 16$

$a_0 = +0.1666666661133027 \times 10^{0}$
$a_1 = -0.8333338509758059 \times 10^{-2}$
$a_2 = +0.1984127524406629 \times 10^{-3}$
$a_3 = -0.2755570214946503 \times 10^{-5}$
$a_4 = +0.2505123071826789 \times 10^{-7}$
$a_5 = -0.1618528418504030 \times 10^{-9}$
$a_6 = +0.7694603615375217 \times 10^{-12}$

$M_a = 0.118 \times 10^{-7}$

$$q_1 = \sqrt[6]{a_6}\, x$$
$$q_2 = (q_1 + A)^2$$
$$q_3 = (q_2 + B)(q_1 + C)$$
$$S(X) = (q_2 + q_3 + D)(+q_3 + E) + F \quad \text{on } [-h, h]$$

$h = 1$

$A = +0.1030541110544949 \times 10^0$
$B = +0.1357446199107850 \times 10^1$
$C = -0.1709888012144409 \times 10^1$
$D = +0.1803960616654583 \times 10^1$
$E = +0.2062171852872241 \times 10^1$
$F = +0.2130010466488949 \times 10^{-1}$

$M = 0.236 \times 10^{-14}$

$h = 2$

$A = +0.1028274494783523 \times 10^0$
$B = +0.1356879505417743 \times 10^1$
$C = -0.1709784631095070 \times 10^1$
$D = +0.1802832347589989 \times 10^1$
$E = +0.2060949727430426 \times 10^1$
$F = +0.2128753997322974 \times 10^{-1}$

$M = 0.808 \times 10^{-14}$

$h = 4$

$A = +0.1026453527945748 \times 10^0$
$B = +0.1356011711587295 \times 10^1$
$C = -0.1709549340846876 \times 10^1$
$D = +0.1800557277079013 \times 10^1$
$E = +0.2059246874696585 \times 10^1$
$F = +0.2125209137884825 \times 10^{-1}$

$M = 0.722 \times 10^{-12}$

$h = 16$

$A = +0.9898746283297480 \times 10^{-1}$
$B = +0.1338586121335949 \times 10^1$
$C = -0.1704756186376375 \times 10^1$
$D = +0.1754906650979839 \times 10^1$
$E = +0.2025050653157090 \times 10^1$
$F = +0.2056593593968585 \times 10^{-1}$

$M = 0.118 \times 10^{-7}$

$$C'(X) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \text{ on } [-h, h]$$

$h = 1$

$a_0 = -0.4166666666666430 \times 10^{-1}$
$a_1 = +0.2777777778467318 \times 10^{-2}$
$a_2 = -0.1488095238678453 \times 10^{-3}$
$a_3 = +0.6613751098214413 \times 10^{-5}$
$a_4 = -0.2505208412532178 \times 10^{-6}$
$a_5 = +0.8269965205281566 \times 10^{-8}$
$a_6 = -0.2412214891589441 \times 10^{-9}$

$M_a = 0.217 \times 10^{-16}$

$h = 2$

$a_0 = -0.4166666666606742 \times 10^{-1}$
$a_1 = +0.2777777822034584 \times 10^{-2}$
$a_2 = -0.1488095275535492 \times 10^{-3}$
$a_3 = +0.6613668137463213 \times 10^{-5}$
$a_4 = -0.2505171918626241 \times 10^{-6}$
$a_5 = +0.8303136595603477 \times 10^{-8}$
$a_6 = -0.2422316681583509 \times 10^{-9}$

$M_a = 0.251 \times 10^{-14}$

$h = 4$

$a_0 = -0.4166666651125364 \times 10^{-1}$
$a_1 = +0.2777780643726382 \times 10^{-2}$
$a_2 = -0.1488097663598733 \times 10^{-3}$
$a_3 = +0.6612326049924104 \times 10^{-5}$
$a_4 = -0.2504581368842600 \times 10^{-6}$
$a_5 = +0.8437138609417991 \times 10^{-8}$
$a_6 = -0.2463133603265637 \times 10^{-9}$

$M_a = 0.320 \times 10^{-12}$

$h = 16$

$a_0 = -0.4166666641763858 \times 10^{-1}$
$a_1 = +0.2777780078584500 \times 10^{-2}$
$a_2 = -0.7440478621862503 \times 10^{-4}$
$a_3 = +0.1102220894089232 \times 10^{-5}$
$a_4 = -0.1043798352532477 \times 10^{-7}$
$a_5 = +0.6938557071056230 \times 10^{-10}$
$a_6 = -0.3366982823031963 \times 10^{-12}$

$M_a = 0.527 \times 10^{-8}$

$$q_1 = \sqrt[6]{-a_6}\ x$$
$$q_2 = (q_1 + A)^2$$
$$q_3 = (q_2 + B)(q_1 + C)$$
$$C'(X) = (q_2 + q_3 + D)(-q_3 - E) - F \quad \text{on } [-h, h]$$

$h = 1$

$A = +0.6421500675794880 \times 10^{-1}$
$B = +0.9631412054424315 \times 10^{0}$
$C = -0.1485378935681960 \times 10^{1}$
$D = +0.1206516480446427 \times 10^{1}$
$E = +0.1262370275488431 \times 10^{1}$
$F = +0.2235785774036898 \times 10^{-2}$

$M = 0.179 \times 10^{-15}$

$h = 2$

$A = +0.6415596248585610 \times 10^{-1}$
$B = +0.9629599640982438 \times 10^{0}$
$C = -0.1485350658374242 \times 10^{1}$
$D = +0.1206153148806730 \times 10^{1}$
$E = +0.1262088821410514 \times 10^{1}$
$F = +0.2230745797490601 \times 10^{-2}$

$M = 0.268 \times 10^{-14}$

$h = 4$

$A = +0.6405925631551870 \times 10^{-1}$
$B = +0.9624584033933687 \times 10^{0}$
$C = -0.1485269967153406 \times 10^{1}$
$D = +0.1205018912730453 \times 10^{1}$
$E = +0.1261394717119364 \times 10^{1}$
$F = +0.2210887038704210 \times 10^{-2}$

$M = 0.320 \times 10^{-12}$

$h = 16$

$A = +0.6208183431484613 \times 10^{-1}$
$B = +0.9523193847619871 \times 10^{0}$
$C = -0.1483582934430130 \times 10^{1}$
$D = +0.1182131124018069 \times 10^{1}$
$E = +0.1247277466997771 \times 10^{1}$
$F = +0.1829570481293727 \times 10^{-2}$

$M = 0.527 \times 10^{-8}$

$$S'(X) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \qquad \text{on } [-h, h]$$

$h = 1$

$a_0 = -0.8333333333333210 \times 10^{-2}$
$a_1 = +0.3968253968616768 \times 10^{-3}$
$a_2 = -0.1653439153716376 \times 10^{-4}$
$a_3 = +0.6012503110063301 \times 10^{-6}$
$a_4 = -0.1927084107177350 \times 10^{-7}$
$a_5 = +0.5511761621292874 \times 10^{-9}$
$a_6 = -0.1418571972378231 \times 10^{-10}$
$M_a = 0.260 \times 10^{-17}$

$h = 2$

$a_0 = -0.8333333333304809 \times 10^{-2}$
$a_1 = +0.3968253991531185 \times 10^{-3}$
$a_2 = -0.1653439171256403 \times 10^{-4}$
$a_3 = +0.6012459474285134 \times 10^{-6}$
$a_4 = -0.1927066737597982 \times 10^{-7}$
$a_5 = +0.5529210296030711 \times 10^{-9}$
$a_6 = -0.1423381311296310 \times 10^{-10}$
$M_a = 0.134 \times 10^{-15}$

$h = 4$

$a_0 = -0.8333333325953303 \times 10^{-2}$
$a_1 = +0.3968255472561213 \times 10^{-3}$
$a_2 = -0.1653440305444520 \times 10^{-4}$
$a_3 = +0.6011754909568150 \times 10^{-6}$
$a_4 = -0.1926786201990815 \times 10^{-7}$
$a_5 = +0.5599576811597955 \times 10^{-9}$
$a_6 = -0.1442776137237285 \times 10^{-10}$
$M_a = 0.168 \times 10^{-13}$

$h = 16$

$a_0 = -0.8333333321480760 \times 10^{-2}$
$a_1 = +0.3968255178485000 \times 10^{-3}$
$a_2 = -0.8267196924462379 \times 10^{-5}$
$a_3 = +0.1002046526767437 \times 10^{-6}$
$a_4 = -0.8029333915166488 \times 10^{-9}$
$a_5 = +0.4617817733427159 \times 10^{-11}$
$a_6 = -0.1978183008506138 \times 10^{-13}$
$M_a = 0.277 \times 10^{-9}$

$$q_1 = \sqrt[6]{-a_6}\; x$$
$$q_2 = (q_1 + A)^2$$
$$q_3 = (q_2 + B)(q_1 + C)$$
$$S'(X) = (q_2 + q_3 + D)(-q_3 - E) - F \quad \text{on } [-h, h]$$

$h = 1$

$A = -0.4272502910304863 \times 10^{-1}$
$B = +0.4460409572215307 \times 10^{0}$
$C = -0.1020276052465536 \times 10^{1}$
$D = +0.4064530281034859 \times 10^{0}$
$E = +0.3450015283629582 \times 10^{0}$
$F = +0.2885054406289098 \times 10^{-2}$

$M = 0.529 \times 10^{-16}$

$h = 2$

$A = -0.4272336986330691 \times 10^{-1}$
$B = +0.4460090638507059 \times 10^{0}$
$C = -0.1020282840851638 \times 10^{1}$
$D = +0.4064007086884238 \times 10^{0}$
$E = +0.3449898120978949 \times 10^{0}$
$F = +0.2883373149053849 \times 10^{-2}$

$M = 0.182 \times 10^{-15}$

$h = 4$

$A = -0.4274056995233864 \times 10^{-1}$
$B = +0.4458621272744412 \times 10^{0}$
$C = -0.1020307887595762 \times 10^{1}$
$D = +0.4061739766937458 \times 10^{0}$
$E = +0.3449170623874602 \times 10^{0}$
$F = +0.2876661053629565 \times 10^{-2}$

$M = 0.169 \times 10^{-13}$

$h = 16$

$A = -0.4310245077956152 \times 10^{-1}$
$B = +0.4428994350437274 \times 10^{0}$
$C = -0.1020789569928175 \times 10^{1}$
$D = +0.4016042135773923 \times 10^{0}$
$E = +0.3434241762739658 \times 10^{0}$
$F = +0.2744481265299470 \times 10^{-2}$

$M = 0.277 \times 10^{-9}$

$$P(X) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

C(X) #4 $[-1, +1]$

$a_0 = +0.5000000000007167 \times 10^0$
$a_1 = -0.4166666601424471 \times 10^{-1}$
$a_2 = +0.1388888879568642 \times 10^{-2}$
$a_3 = -0.2480419696201834 \times 10^{-4}$
$a_4 = +0.2755932664579228 \times 10^{-6}$

$M_a = 0.130 \times 10^{-9}$

S(X) #4 $[-1, +1]$

$a_0 = +0.1666666666667142 \times 10^0$
$a_1 = -0.8333333283147393 \times 10^{-2}$
$a_2 = +0.1984126977913694 \times 10^{-3}$
$a_3 = -0.2755932664326337 \times 10^{-5}$
$a_4 = +0.2505344666229896 \times 10^{-7}$

$M_a = 0.100 \times 10^{-10}$

$$q_1 = Ax$$
$$q_2 = (q_1 + B)^2$$
$$P(x) = (q_1 + q_2 + C)(q_2 + D) + E$$

C(X) #4 $[-1, +1]$

$A = +0.2291221893900772 \times 10^{-1}$
$B = -0.7655416156428518 \times 10^0$
$C = +0.2592589041826887 \times 10^0$
$D = +0.4011554686880556 \times 10^0$
$E = -0.3345008393894142 \times 10^0$

$M = 0.130 \times 10^{-9}$

S(X) #4 $[-1, +1]$

$A = +0.1258104947765253 \times 10^{-1}$
$B = -0.5959855810891701 \times 10^0$
$C = +0.1104585254341674 \times 10^0$
$D = +0.2038526154314646 \times 10^0$
$E = -0.9365973340739070 \times 10^{-1}$

$M = 0.100 \times 10^{-10}$

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and since v and t have the same sign, we have

$$\left| v + \sum_{i=0}^{q} 2^i t \right| \geq |2^{q+1} t| .$$

This simply says that the length of the initial bracketing interval is less than, or equal to, the magnitude of the small end; in turn, this means that the large end of the interval is at most twice the magnitude of the small end.

Now, consider how the endpoint values would be represented in floating-point binary arithmetic (normalized) with an r-bit fraction. If the difference between their binary exponents is at most 1 (which is what we are getting at above), then it can be seen that the number of distinct points in the initial bracketing interval is at most 2^r. Therefore, the number of interval-halving iterations needed—that is, the number of times one reduces his choice of points in the interval by one-half—is at most r. Moreover, it often turns out that f is nearly (or exactly) zero at an end point of one of the half-intervals, so that r iterations are not always needed.

We have treated the special case $|t| \leq |v|$, but we need not restrict ourselves to it. The number of interval-halving iterations needed depends upon the size of t and, if one is willing to iterate a bit more, he can find the initial bracketing interval more quickly by increasing t; the converse of this also holds.

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